A new look at compactness via distances to function spaces

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Universidad de Murcia
http://webs.um.es/beca

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III International Course of Mathematical Analysis Andalucia

The co-authors

- **W. Marciszesky, M. Raja** and B. Cascales, *Distance to spaces of continuous functions*, Topology Appl. **153** (2006), 2303–2319.
- **C. Angosto** and B. Cascales, *Measures of weak noncompactness in Banach spaces*, Topology Appl. (2007).
- **C. Angosto** and B. Cascales, *The quantitative difference between countable compactness and compactness*, Submitted, 2007.
- **C. Angosto, I. Namioka** and B. Cascales, *Distances to spaces of Baire one functions*, Submitted, 2007.

The starting point. . .

List Details

- M. Fabian, P. Hájek, V. Montesinos, and V. Zizler.
  A quantitative version of Krein's Theorem..
  Rev. Mat. Iberoamericana **21** (2005), no. 1, 237–248..
- A. S. Granero.
  An extension of Krein-Šmulian theorem.
  Rev. Mat. Iberoamericana **22** (2005), no. 1, 93–110.
- A. S. Granero, P. Hájek, and V. Montesinos Santalucía.
  Convexity and w^*-compactness in Banach spaces.
  Math. Ann., **328, 4** (2004), 625-631.

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Let E be a Banach space and let $H \subset E$ be a bounded subset of E. Then

$$\widehat{\mathrm{d}}(\overline{\mathrm{co}(H)}, E) \leq 2\widehat{\mathrm{d}}(\overline{H}, E),$$

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- closures are weak*-closures taken in the bidual E^{**};
- $\widehat{\mathrm{d}}(A,E) := \sup\{d(a,E) : a \in A\}$ for $A \subset E^{**}$;
- $\widehat{\mathrm{d}}(A,E) = 0$ iff $A \subset E$. Hence the inequality implies Krein's theorem (if H is relatively weakly compact then $\overline{\mathrm{co}(H)}$ is weakly compact.)

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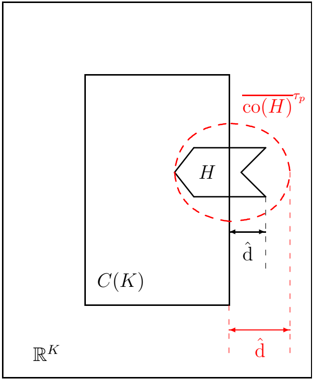
- Some of the constant involved are sharp.

...our goal

...goals

- To take the results where *(I think!)* they belong *i.e.* to the context of $C(K)$ and $\mathbb{R}^K$ spaces endowed with τ_p;

...our goal



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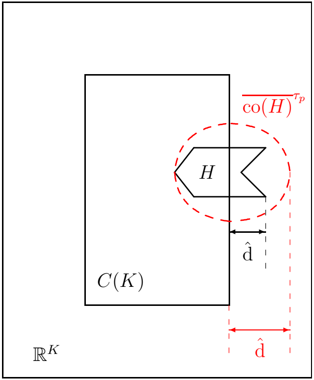

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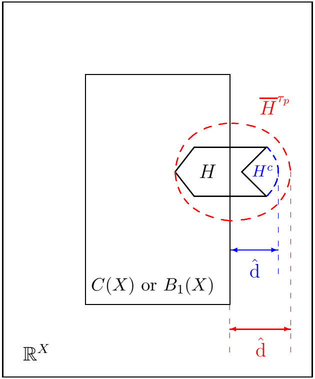

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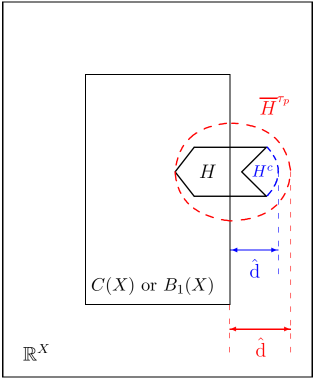

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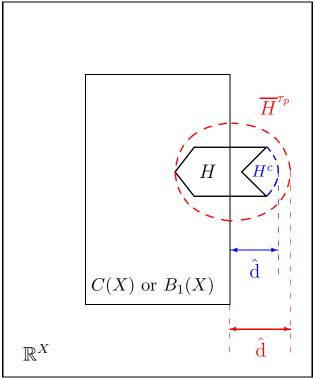

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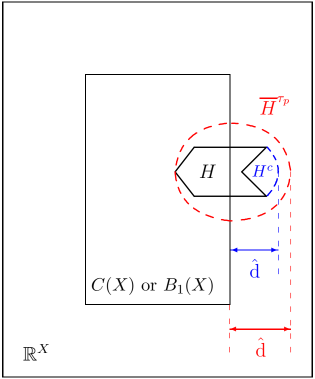

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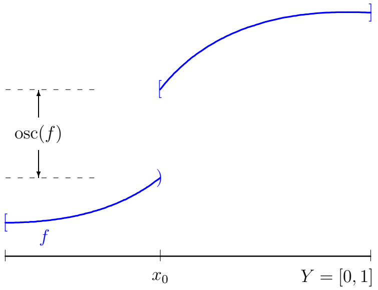
tools

- new reading of the *classical*;
- for $C(X)$ we use *double limits* used by Grothendieck;
- for $B_1(X)$ we use the notions of *fragmentability* and *σ-fragmentability* of functions.

Distances vs. oscillations



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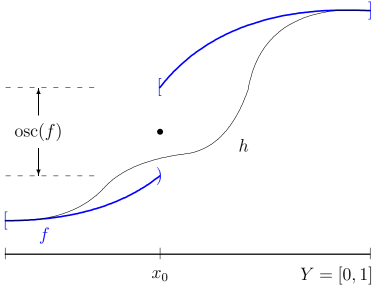

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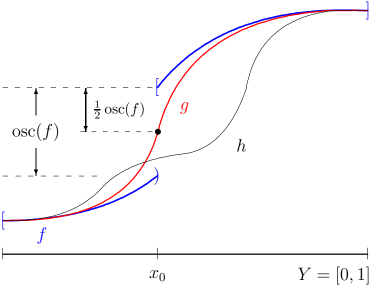

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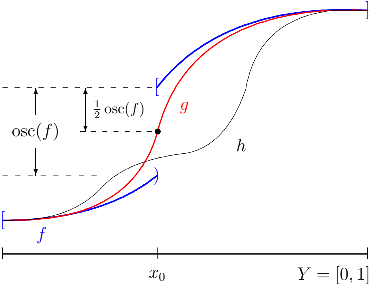

Theorem

Let Y be a normal space [a]. If $f \in \mathbb{R}^Y$ is bounded, then

$$d(f, C_b(Y)) = \frac{1}{2}\operatorname{osc}(f).$$

[a] $[\operatorname{osc}(f) = \sup_{x\in Y} \operatorname{osc}(f,x)]$

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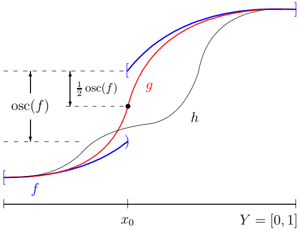

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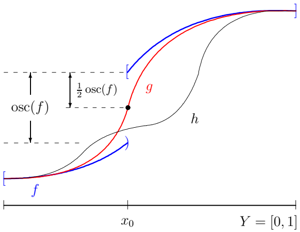

1. It is easy to check that $d(f, C_b(Y)) \geq \operatorname{osc}(f)/2$.

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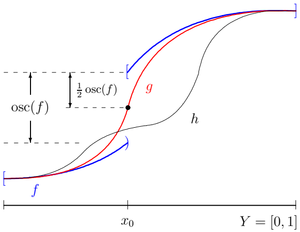

1. It is easy to check that $d(f, C_b(Y)) \geq \operatorname{osc}(f)/2$.
2. For $x \in Y$, $\mathscr{U}_x$ family of neighb.

$$\operatorname{osc}(f) = \inf_{U \in \mathscr{U}_x} \sup_{y,z \in U} \left(f(y) - f(z)\right)$$

$$\geq \inf_{U \in \mathscr{U}_x} \sup_{y \in U} f(y) - \sup_{U \in \mathscr{U}_x} \inf_{z \in U} f(z)$$

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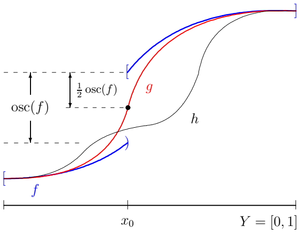

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$$f_2(x) := \sup_{U \in \mathscr{U}_x} \inf_{z \in U} f(z) + \frac{\operatorname{osc}(f)}{2}$$

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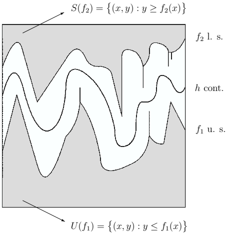

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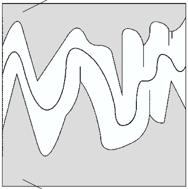

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4. Squeeze h between f_2 and f_1 and $d(f, C_b(Y)) = \|f - h\|_\infty = \operatorname{osc}(f)/2$.

Quantitative Grothendieck charact. of τ_p-compactness

Theorem

If K is a compact topological space and H is a uniformly bounded subset of $C(K)$, then

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If H is relatively countably compact in $C(K)$ then $\operatorname{ck}(H) = 0$

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- in $\gamma(H)$ replace sequences by nets.

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- Take a net in H $(f_\beta) \to f$ in $\mathbb{R}^K$.

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If K is a compact topological space and H is a uniformly bounded subset of $C(K)$, then

$$\mathrm{ck}(H) \overset{(a)}{\leq} \hat{\mathrm{d}}(\overline{H}^{\mathbb{R}^K}, C(K)) \overset{(b)}{\leq} \gamma(H) \overset{(c)}{\leq} 2\,\mathrm{ck}(H).$$

(b)

- in $\gamma(H)$ replace sequences by nets.
- Pick $f \in \overline{H}^{\mathbb{R}^K}$ and fix $x \in K$.
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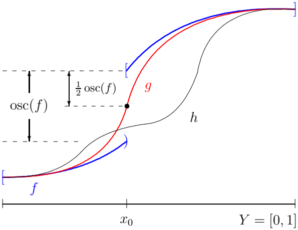

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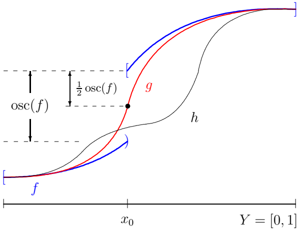

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and in the general case $H \subset \mathbb{R}^K$

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2. When $H \subset \mathbb{R}^K$, we approximate H by some set in $C(K)$, then use (1) and 5 appears as a simple

$$5 = 2 \times 2 + 1.$$

Distances to spaces of affine continuous functions

Theorem

If K is compact convex subset of a l.c.s. and $f \in \mathscr{A}(K)$ then

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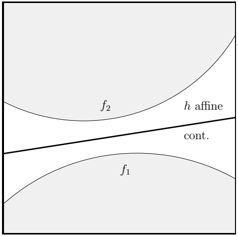

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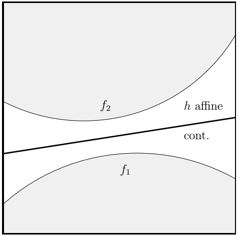

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4. Squeeze h between f_2 and f_1 and $\|f - h\|_\infty \leq \delta/2$.

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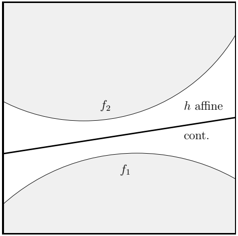

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Corollary

Let X be a Banach space and let B_{X^*} be the closed unit ball in the dual X^* endowed with the w^*-topology. Let $i : X \to X^{**}$ and $j : X^{**} \to \ell_\infty(B_{X^*})$ be the canonical embedding. Then, for every $x^{**} \in X^{**}$ we have:

$$d(x^{**}, i(X)) = d(j(x^{**}), C(B_{X^*})).$$

Measures of weak noncompactness

Definition

Given a bounded subset H of a Banach space E we define:

$$\gamma(H) := \sup\{|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| : (f_m) \subset B_{E^*}, (x_n) \subset H\},$$

assuming the involved limits exist,

$$\mathrm{ck}(H) := \sup_{(h_n)_n \subset H} d\Big(\bigcap_{m\in\mathbb{N}} \overline{\{h_n : n > m\}}^{w^*}, E\Big),$$

$$\mathrm{k}(H) := \hat{d}(\overline{H}^{w^*}, E) = \sup_{x^{**}\in\overline{H}^{w^*}} d(x^{**}, E),$$

where the w^*-closures are taken in E^{**} and the distance d is the usual inf distance for sets associated to the natural norm in E^{**}.

Relationship between measures of weak noncompactness

Theorem

For any bounded subset H of a Banach space E we have:

$$\mathrm{ck}(H) \leq \mathrm{k}(H) \leq \gamma(H) \leq 2\,\mathrm{ck}(H) \leq 2\,\mathrm{k}(H)$$

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For any $x^{**} \in \overline{H}^{w^*}$, there is a sequence $(x_n)_n$ in H such that

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for any cluster point y^{**} of $(x_n)_n$ in E^{**}. Furthermore, H is weakly relatively compact in E if, and only if, it is zero one (equivalently all) of the numbers $\mathrm{ck}(H), \mathrm{k}(H), \gamma(H)$

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The result above is the quantitative version of Eberlein-Smulyan and Krein-Smulyan theorems. From $\mathrm{k}(\mathrm{co}(H)) \le 2\,\mathrm{k}(H)$ straightforwardly follows Krein-Smulyan theorem.

Other applications to Banach spaces

Theorem (C. Angosto, B.C.)

Let K be a compact space and let H be a uniformly bounded subset of $C(K)$. Let us define

$$\gamma_K(H) := \sup\{|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| : (f_m) \subset H, (x_n) \subset K\},$$

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Theorem (C. Angosto, B.C.)

Let E and F be Banach spaces, $T : E \to F$ an operator and $T^* : F^* \to E^*$ its adjoint. Then

$$\gamma(T(B_E)) \leq \gamma(T^*(B_{F^*})) \leq 2\gamma(T(B_E)).$$

Other applications to Banach spaces

Remark: Astala and Tylli [AT90, Theorem 4]

There is separable Banach space E and a sequence $(T_n)_n$ of operators $T_n : E \to c_0$ such that

$$\omega(T_n^*(B_{\ell^1})) = 1 \qquad \text{and} \qquad \omega(T_n^{**}(B_E^{**})) \leq w(T_n(B_E)) \leq \frac{1}{n}.$$

Note that this example says, in particular, that there are no constants $m, M > 0$ such that for any bounded operator $T : E \to F$ we have

$$m\omega(T(B_E)) \leq \omega(T^*(B_{F^*})) \leq M\omega(T(B_E)).$$

Corollary

γ and ω are not equivalent measures of weak noncompactness, namely there is no $N > 0$ such that for any Banach space and any bounded set $H \subset E$ we have

$$\omega(H) \leq N\gamma(H).$$

The results for $C(X)$

If X is a topological space, (Z,d) a metric space and H a relatively compact subset of the space (Z^X,τ_p) we define

$$\operatorname{ck}(H) := \sup_{(h_n)_n\subset H} d\Big(\bigcap_{m\in\mathbb{N}} \overline{\{h_n : n > m\}}^{Z^X}, C(X,Z)\Big).$$

Theorem (C. Angosto, B.C.)

Let X be a countably K-determined space, (Z,d) a separable metric space and H a relatively compact subset of the space (Z^X,τ_p). Then, for any $f\in\overline{H}^{Z^X}$ there exists a sequence $(f_n)_n$ in H such that

$$\sup_{x\in X} d(g(x),f(x)) \overset{(a)}{\leq} 2\operatorname{ck}(H) + 2\hat{d}(H,C(X,Z)) \overset{(b)}{\leq} 4\operatorname{ck}(H)$$

for any cluster point g of (f_n) in Z^X.

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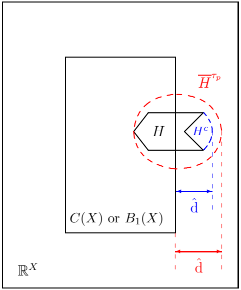
For the particular case $\operatorname{ck}(H)=0$ we obtain all known results about compactness in $C_p(X)$ spaces.

The results for $B_1(X)$. . .



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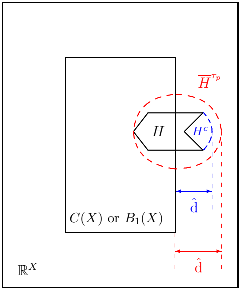

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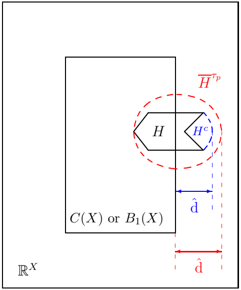

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Definition

If X topological space, (Z,d) a metric and $f \in Z^X$. We define:

$$\operatorname{frag}(f) := \inf\{\varepsilon > 0 : f \text{ is } \varepsilon\text{-fragmented}\}$$

Quantitative version of a Rosenthal's result

Theorem (C. Angosto, I. Namioka and B.C.)

If X is a complete metric space, E a Banach space and $f \in E^X$ then

$$\frac{1}{2}\operatorname{frag}(f) \leq d(f, B_1(X,E)) \leq \operatorname{frag}(f).$$

In the particular case $E = \mathbb{R}$ we precisely have

$$d(f, B_1(X)) = \frac{1}{2}\operatorname{frag}(f).$$

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Theorem (C. Angosto, I. Namioka and B.C.)

If X is a complete metric space, E a Banach space and $f \in E^X$ then

$$\frac{1}{2}\operatorname{frag}(f) \leq d(f, B_1(X,E)) \leq \operatorname{frag}(f).$$

In the particular case $E = \mathbb{R}$ we precisely have

$$d(f, B_1(X)) = \frac{1}{2}\operatorname{frag}(f).$$

Theorem (C. Angosto, I. Namioka and B.C.)

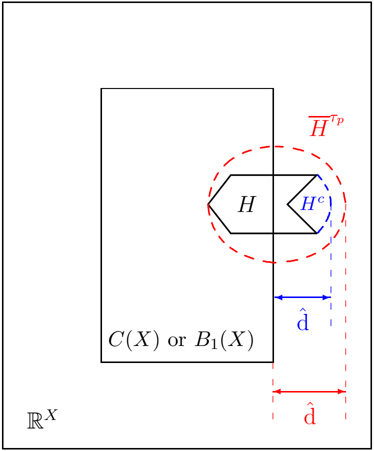
Let X be a Polish space, E a Banach space and H a τ_p-relatively compact subset of E^X. Then

$$\operatorname{ck}(H) \leq \hat{d}(\overline{H}^{E^X}, B_1(X,E)) \leq 2\operatorname{ck}(H).$$

In the particular case when $E = \mathbb{R}$ we have

$$\hat{d}(\overline{H}^{\mathbb{R}^X}, B_1(X)) = \operatorname{ck}(H).$$

Quantitative version of a Rosenthal's result



$\hat{d} = \hat{d}$

Distances to spaces of measurable functions

- (Ω, Σ, μ) is a complete probability space and $(E, \| \ \|)$ is a Banach space.
- $\Sigma^+ = \{B \in \Sigma : \mu(B) > 0\}$ and $\Sigma^+_A = \{B \in \Sigma^+ : B \subset A\}$.
- $M(\mu, E)$ strongly measurable functions from Ω to E.

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Index of strong measurability

Given $f \in E^\Omega$, we define

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Proposition

Let $f \in E^\Omega$. Then:

$$d(f, M(\mu; E)) \leq \text{meas}(f) \leq 2d(f; M(\mu; X)).$$

Moreover, if $E = \mathbb{R}$, then

$$d(f, M(\mu; X)) = \frac{1}{2}\text{meas}(f).$$

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